Stability of relative equilibria of multidimensional rigid body

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Abstract

It is well known that a free three-dimensional rigid body admits three stationary rotations. These are the rotations around three principal axes of inertia. The rotations around the long and the short axes are stable, while the rotation around the intermediate axis is unstable. We generalize this result to the case of a multidimensional body. The stability problem is being solved for a dense subset of stationary rotations in arbitrary dimension.

Contents

1 Introduction

> The above treatment of the problem of rotation may, in contradistinction to the usual method, be transposed, word for word, from three-dimensional space to multidimensional spaces. This is, indeed, irrelevant in practice. On the other hand, the fact that we have freed ourselves from the limitation to a definite dimensional number and that we have formulated physical laws in such a way that the dimensional number appears **accidental** in them, gives us an assurance that we have succeeded fully in grasping them mathematically.
>
> H.Weyl. *Space, Time, Matter.*

1.1 Statement of the problem

Speaking informally, free multidimensional rigid body is simply a rigid body rotating in multidimensional space without action of any external forces (i.e. by inertia).

Let us first discuss a three-dimensional free rigid body (the so-called Euler case in the rigid body dynamics). A good model for such a body is a book or a parallelepiped shaped box.

Throw a book in the air spinning it in arbitrary direction. If we neglect the gravity force, then what we get is exactly the Euler case. Note that a general trajectory of the book is not a rotation in the usual sense. At each moment of time the book is indeed rotating around some axis, but this axis is changing as time goes. What we are interested in, are the relative equilibria of the system, i.e. such trajectories for which the axis of rotation remains fixed. Such rotations are also called stationary.

It is well known that a rotation of a generic three-dimensional rigid body (i.e. a body with pairwise distinct principal moments of inertia) is stationary if and only if the axis of the rotation is one of the three principal axes of inertia. If we deal with a homogeneous parallelepiped shaped body, then the principal axes of inertia coincide with the axes of symmetry.

Let $S > I > L$ be the principal moments of inertia. The corresponding principal axes of inertia are called, respectively, the short, the intermediate, and the long axis of inertia. In the case of a symmetric body these names correspond to the actual lengths of the axes of symmetry (see Figure 1).

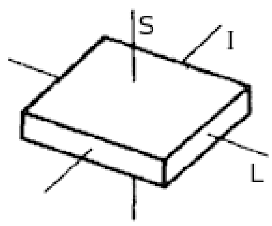


Figure 1: Three-dimensional rigid body.

Stationary rotations around different principal axes of inertia have different dynamical features (see [1–3]). To demonstrate it, spin the body around one of these axes. Of course, since our hands and eyes are not too precise, it is not possible to spin exactly around the chosen axis. But this is not going to be fatal if this axis is the short or the long one. We will not have a stationary rotation, but something very close to it. This is due to the fact that *the rotations of a free three-dimensional rigid body around the short and the long principal axes of inertia are (Lyapunov) stable.* But if we spin around the intermediate axis, we will see something essentially different: the axis of rotation will start changing rapidly and the body will start rotating in other direction. This is because *the rotation of a free three-dimensional rigid body around the intermediate principal axis of inertia is (Lyapunov) unstable.*

Basically, what we are interested in, is to generalize this result to the case of a multidimensional body.

Remark 1.1. The system describing the motion of a free multidimensional rigid body (see Section 1.3) is also known as the Manakov top. Manakov's achievement was to prove that this system is integrable (see [4]). However, the equations themselves go back to Frahm [5]. In the famous paper [6] Arnold wrote these equations in the form of hamiltonian equations on $\mathfrak{so}(n)^*$ and generalized them to the case of an arbitrary Lie algebra (see also [2]). A possibility to generalize the equations of a free rigid body to the multidimensional case was also mentioned by Weyl [7].

First we shall discuss how an n-dimensional body may rotate. At each moment of time $\mathbb{R}^n$ is decomposed into a sum of m pairwise orthogonal two-dimensional planes $\Pi_1, \dots, \Pi_m$ and an $n-2m$-dimensional space Π_0 orthogonal to all these planes:

$$\mathbb{R}^n = \left(\bigoplus_{i=1}^{m} \Pi_i \right) \oplus \Pi_0.$$

There is an independent rotation in each of the planes $\Pi_1, \dots, \Pi_m$, while Π_0 is fixed. This is just a reformulation of the theorem about canonical form of a skew-symmetric operator. Note that Π_0 may be zero in the even-dimensional case, which means that there are no fixed axes.

A rotation is stationary if all the planes $\Pi_0, \dots, \Pi_m$ don't change with time (this condition automatically implies that the velocities of the rotations are also constant).

Before studying stationary rotations for stability, we need to find these rotations. Recall that a rotation of a generic three-dimensional rigid body is stationary if and only if it is a rotation around one of the principal axes of inertia. In the multidimensional case the situation is slightly more complicated. If the planes $\Pi_0, \dots, \Pi_m$ are spanned by principal axes of inertia (such rotations are called in [8] *regular*), then the rotation is stationary. But the converse is not necessarily true (see [8] and Section 1.5 of the present paper). However, it is true provided that the angular velocities of the rotations in the planes $\Pi_1, \dots \Pi_m$ are pairwise distinct (see [9, 10]). Moreover, we will show that the rotations for which the planes $\Pi_0, \dots, \Pi_m$ are not spanned by principal axes of inertia (such rotations are called in [8] *exotic*) are always unstable. Therefore, we will mainly deal with regular stationary rotations.

In four dimensions the problem of stability was studied from different points of view by Oshemkov [11], Feher and Marshall [9], Birtea, Caşu, Ratiu, and Turhan [10], Birtea and Caşu [12]. The answer is known for a dense subset of relative equilibria.

Remark 1.2. Oshemkov didn't actually pose the stability problem. However, the answer can be easily obtained from his bifurcation diagrams.

The five-dimensional case was studied by Caşu in [13]. The set of equilibria which were studied in this case is not dense.

General even-dimensional rigid body was discussed in Spiegler's PhD thesis [14]. A sufficient condition for an equilibrium to be stable is found.

In this paper we solve the stability problem for a dense subset of relative equilibria in arbitrary dimension. This result is presented in Section 2.

Recall that in three dimensions stability depends only on the rotation axis. In many dimensions the situation is slightly more complicated: angular velocities of rotations may also affect stability. However, the answer is still rather elegant, and the classical three-dimensional result may be viewed as a particular case of this general answer.

1.2 Methods of stability investigation

In this section we will discuss different approaches to the problem of stability of relative equilibria of a multidimensional rigid body.

Spiegler [14] approached the problem using the method known as Arnold or *energy-Casimir method* (the method was introduced in [6]; see also [2]). This method can be formulated as follows: Let x be an equilibrium point of a hamiltonian system on a Poisson manifold. Then x is a critical point for the restriction of the Hamiltonian to the symplectic leaf O passing through x. If this critical point is a non-degenerate minimum or maximum, and O is regular, then x is stable.

The energy-Casimir method is a very powerful tool for studying equilibria of general Hamiltonian systems (especially inifinite-dimensional, see [3, 15–17] for an overview). But if a system possesses some additional symmetries (for example, it is completely integrable), there are more efficient ways to study stability. Let us consider the following simple example: take the Hamiltonian

$$H = \sum_{i=1}^{n} \omega_i (p_i^2 + q_i^2). \tag{1}$$

If there are ω_i's of different sign (which is very likely if n is large), then the origin is not a minimum or a maximum point for H. Consequently, we can't apply the energy-Casimir method. But if we observe that the system possesses an integral

$$f = \sum_{i=1}^{n} p_i^2 + q_i^2,$$

then stability is immediately established.

This example shows what normally happens in integrable systems. Since the Birkhoff normal form for an integrable system always converges (see [18, 19]), the Hamiltonian can be brought to the form (1) in the neighborhood of a generic linearly stable equilibrium. Since ω_i's are, in general, of different signs, the energy-Casimir method in the integrable case does not work for "most" stable equilibria.

As it was shown by Manakov in [4], the system describing the motion of a multidimensional rigid body is integrable. Therefore we may apply methods developed in the theory of integrable hamiltonian systems.

Remark 1.3. More precisely, Manakov showed that the system admits an $L - A$ pair with a spectral parameter, which allowed him to find integrals. However, he did not prove these integrals are enough for Liouville integrability. Complete integrability in the Liouville sense was proved by Fomenko and Mischenko in [20] and by Ratiu in [21]. See also earlier paper [22], where the quadratic integrals of the problem are found, and [23], where these integrals are proved to be in involution.

Manakov also showed that the system is integrable in θ-functions of Riemann surfaces (see also [24], where the four-dimensional case is integrated by elementary means).

Let us mention two possible ways to study stability of equilibria of an integrable hamiltonian system:

1) *Extended energy-Casimir method.* The idea is to replace the Hamiltonian with any other integral in the formulation of the energy-Casimir method. The extended method can be

formulated as follows: Let x be an equilibrium point of a hamiltonian system on a Poisson manifold, and let O be the symplectic leaf passing through x. Suppose that the system possesses integrals $f_1, \dots f_n$. Assume that we can find a linear combination $f = \sum c_i f_i$, such that the restriction of f to O has a non-degenerate minimum at the point x. Then, if O is regular, x is stable.

Remark 1.4. Note that if x is not critical for the restriction of f to O, then the whole trajectory of the hamiltonian flow generated by f passing through x consists of equilibrium points. In this situation it is possible to show that x is unstable, provided the system is non-resonant.

This method was applied in four dimensions in [9, 12]. In five dimensions it was applied in [13], however only quadratic integrals were considered (while this system possesses also a cubic integral).

2) *Eliasson theorem* (this theorem is actually due to Russmann [25], Vey [26], Eliasson [27, 28] and Miranda [29]). The theorem provides a normal form for the momentum mapping in the neighborhood of any equilibrium satisfying the so-called non-degeneracy condition. There is only a finite number of normal forms. They are distinguished by the so-called *Williamson type* of a point. Only one type corresponds to stable behavior. Therefore, the stability problem is easily solved provided we can verify non-degeneracy and determine the type of a point. See [30] for precise defintions.

The disadvantage of this method is that it works only for non-degenerate points. Provided the non-degeneracy condition is satisfied, the method is equivalent to the extended energy-Casimir method. However, if it is not satisfied, it is still possible that the energy-Casimir method works, while the Eliasson theorem does not.

The advantage of the method is that we get much more information, than simply stability/instability.

This method was applied in four dimensions in [10]. However, all the necessary information can be extracted from the bifurcation diagrams constructed in [11].

Both methods are very efficient, but their explicit application involves heavy calculations if the dimension is larger then four, because the number and the degree of integrals grow with the dimension. This brings implicit methods to the stage. One of such methods is to make use of the presence of a bihamiltonian structure.

Remark 1.5. The notion of a bihamiltonian system was introduced by F. Magri in [31]. Bihamiltonian structure is known to be closely related with integrability. On the one hand, if a system is bihamiltonian, then it admits “many” integrals in involution (see [31–34]). On the other hand, many integrable equations coming from mechanics, mathematical physics and geometry possess a bihamiltonian structure (see, for example, [31, 35–43]).

The bihamiltonian structure for the multidimensional rigid body equations was discovered by A.Bolsinov in [44] (see also [45–47]). This structure is defined on the dual space of the Lie algebra of skew-symmetric matrices.

Remark 1.6. However, it is possible to give another bihamiltonian formulation: the bihamiltonian structure is defined on the dual of $\mathfrak{sl}(n)^*$ and then the system is obtained by the restriction from $\mathfrak{sl}(n)^*$ to $\mathfrak{so}(n)^*$. This structure is implicitly present in [20] and explicitly written in [48]. See also [21].

The bihamiltonian approach for studying relative equilibria of a multidimensional rigid body was introduced in [49]. In this paper (among other results) Bolsinov and Oshemkov obtain a sufficient condition for non-degeneracy. Developing their ideas and applying the technique of [50], we give a necessary and sufficient condition for an equilibrium to be non-degenerate and determine type of all non-degenerate points. This allows us to solve the stability problem for a dense subset of relative equilibria.

1.3 The equations

The motion of a free multidimensional rigid body is described by the Euler-Arnold (or Euler-Frahm) equations on $\mathfrak{so}(n)^*$ (identified with $\mathfrak{so}(n)$). These equations have the form

$$\begin{cases} \dot{M} & = [M, \Omega] \\ M & = \Omega J + J\Omega, \end{cases} \tag{2}$$

where

- $M \in \mathfrak{so}(n)^*$ is a skew-symmetric matrix, called the angular momentum matrix.
- J is symmetric. We call it the covariance matrix of the body (see Section 1.4).
- Ω is a skew-symmetric matrix, called the angular velocity matrix. It is uniquely defined by the relation
$$M = \Omega J + J\Omega.$$

For explicit derivation of these equations see [21].

Remark 1.7. Since the map $\mathcal{J} \colon \mathfrak{so}(n) \to \mathfrak{so}(n)$ given by the formula

$$\mathcal{J}(\Omega) = \Omega J + J\Omega \tag{3}$$

is invertible, our equations can be rewritten in the Ω-coordinates:

$$\dot{\Omega} = \mathcal{J}^{-1}([\mathcal{J}(\Omega), \Omega]).$$

However, the explicit formula for $\mathcal{J}^{-1}$ is complicated, therefore it is convenient to introduce the variable M and write down the equations in the form (2).

Remark 1.8. Note that the equations (2) describe only the dynamics of the angular velocity matrix. If we want to recover the dynamics in the whole phase space $\mathrm{T}^*\mathrm{SO}(n)$, we should add Poisson equations

$$\dot{X} = X\Omega.$$

However, we will only be interested in the reduced dynamics, given by the system (2). Note that relative equilibria of a rigid body are nothing else but the equilibrium points of the system (2).

1.4 The inertia tensor versus the covariance matrix

In the multidimensional case the matrix J entering the equations (2) is sometimes referred as the "inertia tensor". This is not very precise, because in the three-dimensional case the inertia tensor is not J, but the map $\mathcal{J} \colon \mathfrak{so}(3) \to \mathfrak{so}(3)$ given by the formula (3). The coincidence of dimensions of $\mathbb{R}^3$ and $\mathfrak{so}(3)$ makes it possible to consider both J and $\mathcal{J}$ as operators (or bilinear forms) on $\mathbb{R}^3$. In the multidimensional case these operators act on different spaces. We suggest the following definitions:

- The *inertia tensor* is $\mathcal{J}$. J is the *covariance matrix (tensor)* of the body. Recall (see [21]) that the entries of J are the values of the covariance of the coordinates:
$$J_{ij} = \int (x_i - \widehat{x}_i)(x_j - \widehat{x}_j)\mathrm{d}\mu,$$
where $\widehat{x}_i$ are the center of mass coordinates.
- The *principal moments of inertia* are the eigenvalues of the inertia tensor $\mathcal{J}$. If we denote the eigenvalues of the covariance matrix J by λ_i, then the principal moments of inertia are $I_{ij} = \lambda_i + \lambda_j$ for $i \neq j$.

- The *principal axes of inertia* are the eigenvectors of the covariance matrix. If we denote them by e_i, the eigenvectors of the inertia tensor $\mathcal{J}$ will be $e_i \wedge e_j$ for $i \neq j$. Note that in the three-dimensional case the eigenvectors of the inertia tensor and the covariance matrix coincide.
- A body is *generic* if the eigenvalues of the covariance matrix are pairwise distinct. If all the principal moments of inertia are pairwise distinct, then the body is generic. The converse is not necessarily true if the dimension is larger than three. In three dimensions these two statements are equivalent.

1.5 Description of relative equilibria

Here we briefly state the results of [8].

Theorem 1. *Consider the system of Euler-Arnold equations (2). Suppose that J has pairwise distinct eigenvalues. Then M is an equilibrium point of the system if and only if there exists an orthonormal basis such that J is diagonal, and Ω is block-diagonal of the following form*

$$\Omega = \begin{pmatrix} \omega_1 A_1 & & & & \\ & \ddots & & & \\ & & \omega_k A_k & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix}, \tag{4}$$

where $A_i \in \mathfrak{so}(2m_i) \cap \mathrm{SO}(2m_i)$ for some $m_i > 0$, and ω_i's are distinct positive real numbers.

Corollary 1.1. *Suppose that M is a relative equilibrium, all eigenvalues of J are pairwise distinct. Moreover, let all non-zero eigenvalues of Ω (eigenfrequencies of rotation) be pairwise distinct. Then there exists an orthonormal basis such that J is diagonal, while Ω and M are block-diagonal with two-by-two blocks on the diagonal.*

In other words, a stationary rotation with pairwise distinct eigenfrequencies is a rotation "in principal axes of inertia".

Definition 1. We will say that an equilibrium M is *regular* if there exists an orthonormal basis such that J is diagonal and Ω is block-diagonal with two-by-two blocks on the diagonal (i.e. M is a rotation "in principal axes of inertia"). Otherwise, we will say that M is *exotic.*

Corollary 1.1 says that all stationary rotations with pairwise distinct eigenfrequencies are regular.

We will see later that regular equilibria are exactly rank zero singular points of the system in the sense of integrable systems theory, which means that these points are critical points for *all* integrals. Exotic equilibria are, on the contrary, non-zero rank singular points, which means that there exists an integral the differential of which doesn't vanish in the equilibrium point. This implies that exotic equilibria are not isolated on a coadjoint orbit, but form whole smooth submanifolds of equilibrium points, while regular equilibria are always isolated (on a coadjoint orbit).

Remark 1.9. However, it is possible for a regular equilibrium to belong to the closure of the set of exotic equilibria.

We will also show that exotic equilibria are always unstable.

2 Stability

2.1 Parabolic diagram of a regular relative equilibrium

Suppose we have a regular relative equilibrium. Then there exists an orthonormal basis such that J is diagonal and Ω is block-diagonal with two-by-two blocks on the diagonal. In other words, there exists a decomposition

$$\mathbb{R}^n = \left(\bigoplus_{i=1}^{m} \Pi_i \right) \oplus \Pi_0, \tag{5}$$

where

- Each $\Pi_i, i > 0$ is spanned by two principal axes of inertia. There is a rotation in each of these planes with angular velocity ω_i.
- Π_0 is spanned by $n - 2m$ principal axes of inertia and is fixed.

Therefore, to define a regular relative equilibrium, we need to choose an integer m such that $0 \leq m \leq [n/2]$ and pick m pairs out of the set of the principal axes of inertia. For each pair we need to define an angular velocity. Choice of pairs and values of angular velocities uniquely define a regular relative equilibrium. Knowing this data, we want to understand whether an equilibrium is stable or not.

We are going to define an object called *the parabolic diagram* of a regular relative equilibrium. This object will allow us to express stability conditions in geometric terms.

For each plane $\Pi_i, i > 0$ let us denote by $\lambda_1(\Pi_i), \lambda_2(\Pi_i)$ the eigenvalues of the covariance matrix J corresponding to the principal axes of inertia which span Π_i. By $\omega(\Pi_i)$ denote the angular velocity of rotation in the plane Π_i.

Definition 2. *The parabolic diagram* of a regular relative equilibrium is the following set of parabolas and vertical lines drawn on the same coordinate plane:

- For each Π_i draw a parabola given by

$$\chi_i(x) = \frac{(x - \lambda_1(\Pi_i)^2)(x - \lambda_2(\Pi_i)^2)}{\omega(\Pi_i)^2(\lambda_1(\Pi_i) + \lambda_2(\Pi_i))^2}. \tag{6}$$

- For all fixed principal axes draw vertical lines through the squares of corresponding eigenvalues of J.

Remark 2.1. Each χ_i is a quadratic function the roots of which are the squares of the eigenvalues of J. Therefore what we do is we simply draw parabolas through the squares of eigenvalues corresponding to moving axes and vertical lines through the squares of eigenvalues corresponding to fixed axes.

Remark 2.2. The leading coefficient of χ_i is inverse proportional to the square of the angular momentum $m(\Pi_i) = \omega(\Pi_i)(\lambda_1(\Pi_i) + \lambda_2(\Pi_i))$.

Let us accept the following formal agreement:

- Two parabolas intersect at infinity, if their leading coefficients are equal, i.e. if they have only one point of intersection (of multiplicity one) or no points of intersection (neither real, nor complex).
- Two parabolas are tangent at infinity if they have no points of intersection (real or complex), i.e. if they can be obtained from each other by a vertical shift. Note that this is only possible if the covariance matrix eigenvalues satisfy $\lambda_i^2 + \lambda_j^2 = \lambda_k^2 + \lambda_l^2$ for some distinct i, j, k, l.

After accepting this agreement, the following becomes true:

Proposition 2.1. *Any two parabolas on a parabolic diagram intersect exactly at two points with multiplicities.*

Definition 3. We will say that a parabolic diagram is generic if all intersections on it are simple, i.e. it contains no multiple intersections and no points of tangency.

2.2 Stability theorems

The following three theorems are the main results of the paper.

Theorem 2. *Let a generic multidimensional rigid body be in a state of regular relative equilibrium. Assume that*

- *All intersections on the parabolic diagram of the equilibrium are either real and belong to the upper half-plane or infinite.*
- *The parabolic diagram is generic.*
- *The equilibrium has no more than two fixed axes.*

Then M is stable.

Theorem 3. *Let a generic multidimensional rigid body be in a state of regular relative equilibrium. Assume that*

- *There is at least one intersection on the parabolic diagram of the equilibrium which is either complex or belong to the lower half-plane.*
- *The equilibrium has no more than two fixed axes.*

Then M is unstable.

Theorem 4. *All exotic relative equilibria of a generic multidimensional rigid body are unstable.*

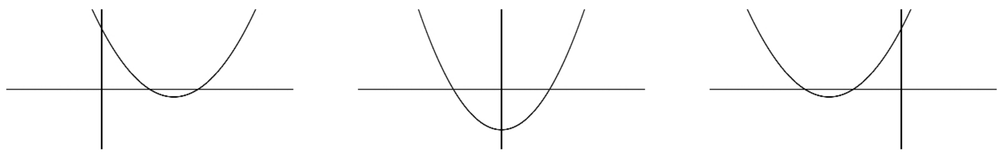
Conjecture (Stability criteria)**.** *A relative equilibrium is stable if and only if it is regular and all intersections on its parabolic diagram are either real and belong to the upper half-plane or infinite.*

Whether or not this conjecture is true, Theorems 2, 3, 4 are already enough to solve the stability problem for an open dense subset of relative equilibria. Proof of these three theorems can be found in Section 5.

2.3 Examples

Now we shall discuss some examples. First let us recover the classical three-dimensional result.

Example 2.1 (Three-dimensional body)**.** Parabolic diagrams for a 3-dimensional body are illustrated on Figure 2. We see from the parabolic diagrams that the rotation around the intermediate axis (diagram 2) is unstable, while the rotations around the short (diagram 1) and the long (diagram 3) axes are stable.



1. Short axis: stable 2. Intermediate axis: unstable 3. Long axis: stable

Figure 2: Parabolic diagrams for a 3d-body

Example 2.2 (Four-dimensional body)**.** Parabolic diagrams for a 4-dimensional body are illustrated on Figure 3. We assume that $\lambda_1^2 + \lambda_4^2 \neq \lambda_2^2 + \lambda_3^2$, where $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$ are the eigenvalues of the covariance matrix.

A four-dimensional body has three different modes of rotation:

1. The first plane of rotation is spanned by two short axes of inertia, the second plane is spanned by two long axes of inertia. We look at the parabolic diagrams (see diagrams 1,2) and see that such rotation is stable.
2. The first plane of rotation is spanned by the shortest and the second longest axes, the second plane is spanned by the longest and the third shortest axes. We look at the parabolic diagrams (see diagrams 3,4) and see that such rotation is unstable.
3. The first plane of rotation is spanned by the longest and the shortest axes, and the second plane is spanned by two intermediate axes. See diagrams 5-11. We see that stability depends in this case on the ratio of angular velocities. If the rotation in the "inner" (i.e. spanned by the intermediate axes) plane is fast enough, we have stability. If it is slow, we have instability. The situation is similar to the rotation of a $3d$ body with a gyroscope inside. If the gyroscope is rotating fast enough, it stabilizes the rotation of the body around the intermediate axis of inertia. The "inner" plane in a four-dimensional body plays the role of a gyroscope.

   In the non-generic case illustrated on diagram 7 our theorems do not give an answer. However, it is possible to show that the corresponding rotation is stable. This implies that stability loss in a four-dimensional body is always "soft", i.e. the set of stable equilibria is closed.

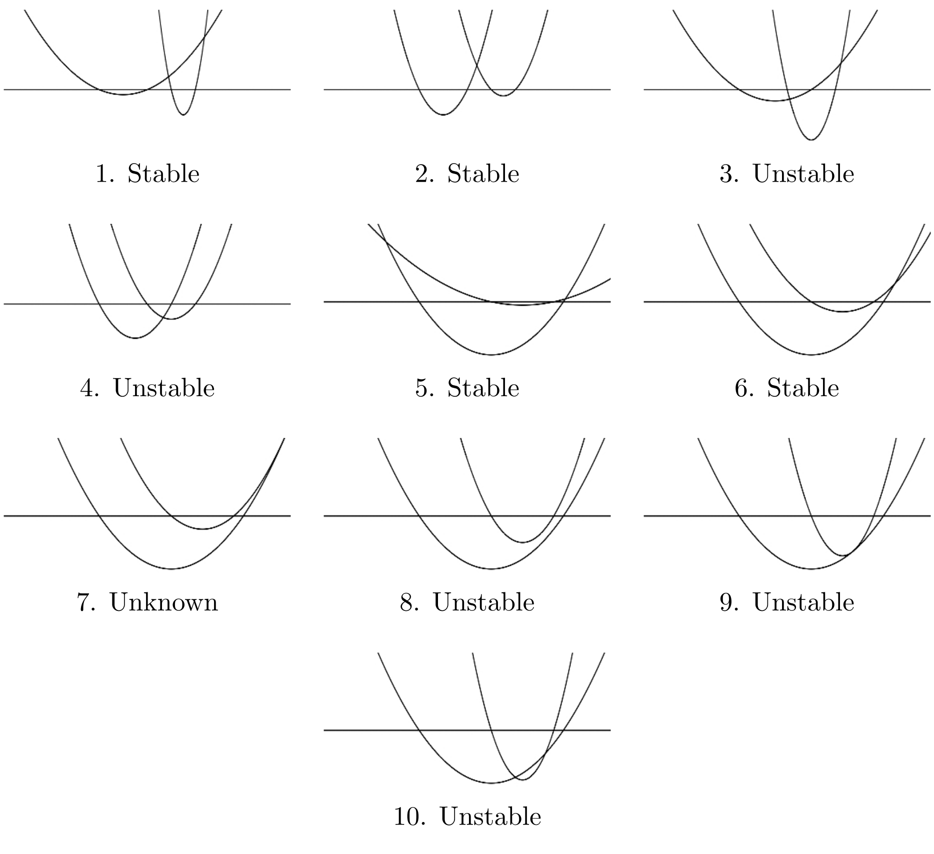


Figure 3: Parabolic diagrams for a 4d-body

3 Bihamiltonian structure and complete integrability

3.1 The bihamiltonian structure

We will denote the standard Lie bracket on $\mathfrak{so}(n)$ by $[\,,]_\infty$ and the corresponding Lie-Poisson bracket on $\mathfrak{so}(n)^*$ by $\{\,,\}_\infty$. It is given by

$$\{f,g\}_\infty(M) = \langle M, [\mathrm{d}f, \mathrm{d}g]_\infty\rangle, \text{ where } M \in \mathfrak{so}(n)^*.$$

The following is standard

Proposition 3.1. *The equations (2) are hamiltonian with respect to the Lie-Poisson bracket $\{\,,\}_\infty$. The Hamiltonian is given by the kinetic energy*

$$H = \frac{1}{2}\langle \Omega, M\rangle$$

Now we introduce a second operation on $\mathfrak{so}(n)$ defined by

$$[X,Y]_0 = XJ^2Y - YJ^2X.$$

Proposition 3.2.

1. *$[\,,]_0$ is a Lie bracket compatible with the standard Lie bracket. In other words, any linear combinations of these brackets define a Lie algebra structure on $\mathfrak{so}(n)$.*
2. *The corresponding Lie-Poisson bracket $\{\,,\}_0$ on $\mathfrak{so}(n)^*$ given by*

$$\{f,g\}_0 = \langle M, [\mathrm{d}f, \mathrm{d}g]_0\rangle.$$

is compatible with the Lie-Poisson bracket $\{\,,\}_\infty$.

Consequently, we have a Lie pencil on $\mathfrak{so}(n)$ and a Poisson pencil on $\mathfrak{so}(n)^*$. We will write down these pencils in the form

$$[X,Y]_\lambda = [X,Y]_0 - \lambda[X,Y]_\infty = X(J^2 - \lambda E)Y - Y(J^2 - \lambda E)X, \tag{7}$$

$$\{f,g\}_\lambda = \{f,g\}_0 - \lambda\{f,g\}_\infty = \langle M, \mathrm{d}f(J^2 - \lambda E)\mathrm{d}g - \mathrm{d}g(J^2 - \lambda E)\mathrm{d}f\rangle. \tag{8}$$

Theorem 5 (A.Bolsinov, [44–46])**.** *The system (2) is hamiltonian with respect to any bracket $\{\,,\}_\lambda$, i.e. it is bihamiltonian. The hamiltonian is given by*

$$H_\lambda = -\frac{1}{2}\langle (J + \sqrt{\lambda}E)^{-1}\Omega(J + \sqrt{\lambda}E)^{-1}, M\rangle. \tag{9}$$

Remark 3.1. Since J is positive-definite, the matrix $J + \sqrt{\lambda}E$ is invertible for any real λ. Note that for negative λ the function H_λ is complex. If we want a real hamiltonian, we must take the real part of H_λ (while the complex part is a Casimir function).

The formula for H_λ given here is different from the one in [44–46]. The difference is a Casimir function.

3.2 The bad set

Since we know that our system is bihamiltonian, we can apply the construction of [50]. According to this construction, the first thing we should do is to describe the set *Bad*, i.e. the set of points in which the rank of all brackets drops. This is the set where the construction of [50] does not work.

Proposition 3.3. *Let M be a regular relative equilibrium. Then $M \in Bad$ if and only if $\dim \operatorname{Ker} M > 2$.*

The proof can be found in Section 4.4. The construction of Section 1.4 of [50] allows us to obtain an involutive system of integrals in the neighborhood of any point $M \notin Bad$. But in our case there are globally defined integrals (because Casimir functions of all brackets of the pencil are globally defined). It is easy to check that the global system of integrals locally coincides with the local one (which is not always the case, see Example 2.4 in [50]). Therefore, it is possible to apply the theorems of [50] to these global integrals.

However, we still cannot say anything about the points which belong to the *Bad* set (though our global integrals are defined on this set as well).

3.3 Complete integrability

Liouville integrability of Euler-Arnold equations was proved in [20]. This can also be easily done using Bolsinov criteria [44] of integrability of bihamiltonian systems. In both cases integrability is established on almost all coadjoint orbits. If we want to prove integrability on a *given* coadjoint orbit, we may apply another argument:

Proposition 3.4. *Let $\mathcal{F}$ be an analytic involutive system on a symplectic manifold M^{2n}. Suppose there exists a non-degenerate singular point of $\mathcal{F}$. Then $\mathcal{F}$ is complete on M^{2n}.*

This may sound strange, because non-degeneracy is usually defined for systems which are already known to be completely integrable. But actually the definition of non-degeneracy works for any involutive system of integrals. And if the condition of non-degeneracy is satisfied, then the system of integrals is automatically complete.

We will give a simple criteria (see Section 4.1) which makes it possible to check non-degeneracy of a zero rank singular point. Therefore, we have a simple sufficient condition for completeness on a given coadjoint orbit.

4 Non-degeneracy

4.1 Non-degeneracy and type theorems

Recall that a zero rank singular point of an integrable system on a symplectic manifold is a point in which the differentials of all integrals vanish.

Let $M \notin Bad$. Take any bracket P_α of the pencil (8) such that $\operatorname{rank} P_\alpha(M)$ is equal to the rank of the pencil. Let $O(M, \alpha)$ be the symplectic leaf of this bracket passing through the point M.

Theorem 6 (Bolsinov, Oshemkov [49])**.** *$M \notin Bad$ is a zero-rank singular point for the system restricted to $O(M, \alpha)$ if and only if there exists an orthonormal basis such that J is diagonal and M is block-diagonal with two-by-two blocks on the diagonal.*

In other words $M \notin Bad$ is a zero-rank singular point if and only if it is a regular equilibrium point.

The following theorem gives a condition for non-degeneracy of a zero rank singular point.

Theorem 7. *A zero-rank singular point $M \notin Bad$ is non-degenerate for the system restricted to $O(M, \alpha)$ if and only if the parabolic diagram of M is generic.*

Remark 4.1. Note that α is chosen in such a way that $\operatorname{rank} P_\alpha(M)$ is equal to the rank of the pencil. If this is not the case, it is possible for M to be non-degenerate even if the parabolic diagram is not generic. For example consider diagram 7 on Figure 3. The corresponding singular point is degenerate on all four-dimensional symplectic leafs passing through it. However it is non-degenerate elliptic on the singular two-dimensional symplectic leaf. This argument can be used for proving stability of the point.

Remark 4.2. A sufficient condition for non-degeneracy is given in [49]. In our terms it means the following:

- The parabolic diagram is generic.
- For each λ there is no more than one intersection point on the parabolic diagram with x coordinate equal to λ.

Theorem 8. *The Williamson type of a non-degenerate zero-rank singular point $M \notin Bad$ is (k_e, k_h, k_f), where*

- *k_e is the number of real intersections on the parabolic diagram in the upper half-plane plus the number of intersections at infinity,*
- *k_h is the number of real intersections in the lower half-plane,*
- *k_f is half the number of complex intersections.*

Scheme of the proof of Theorems 7, 8.

According to the general scheme of [50] to prove non-degeneracy and find the type of a singular point M we should do the following:

- Find those λ, for which the rank of $P_\lambda(M)$ drops, i.e. describe the set $\Lambda(M)$. This is done in Section 4.4.
- Check that the pencil is diagonalizable at M. This is done in Section 4.5.
- Linearize the pencil, check that linearizations are non-degenerate and find their type. This is done in Section 4.6.
- Collect all this together. This is done in Section 4.7.

4.2 Examples

Now we shall discuss examples.

Example 4.1 (Three-dimensional body)**.** Parabolic diagrams for a 3-dimensional body and are illustrated on Figure 2. Rotations around the long and the short axes are elliptic singular points (diagrams 1,3). Rotation around the intermediate axis is a hyperbolic singularity (diagram 2).

Example 4.2 (Four-dimensional body)**.** Parabolic diagrams for a 4-dimensional body are illustrated on Figure 3. The first case from Example 2.2 corresponds to a center-center singular point (see diagrams 1,2). The second case corresponds to a center-saddle singular point (see diagrams 3,4). In the third case everything depends on the ratio of angular velocities. As we change the angular velocity of rotation in the “inner” plane, the following bifurcations occurs:

Center-center (diagrams 5,6) $\to$ degenerate (diagram 7) $\to$ focus-focus (diagram 8) $\to$ degenerate (diagram 9) $\to$ saddle-saddle (diagram 10).

4.3 Notations

Let us fix the notations which will be used throughout the proofs. We'll mainly discuss regular equilibria (except for the proof of Theorem 4). Therefore, we may always assume that there exists an orthonormal basis such that J is diagonal, while Ω and M are block-diagonal with two-by-two blocks on the diagonal.

Let us denote by λ_i the diagonal elements of J **in this basis**. Note that this means that λ_i are possibly different for different equilibria. However, they are unique up to a permutation and coincide with the eigenvalues of J.

By ω_i's we will denote the non-zero entries of matrix Ω, i.e.

$$\Omega = \begin{pmatrix} 0 & \omega_1 & & & & & & \\ -\omega_1 & 0 & & & & & & \\ & & \ddots & & & & & \\ & & & 0 & \omega_m & & & \\ & & & -\omega_m & 0 & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{pmatrix}$$

By $m_i = (\lambda_{2i-1} + \lambda_{2i})\omega_i$ we will denote the entries of the matrix M. M_i will stand for diagonal two-by-two blocks of M i.e.

$$M_i = \begin{pmatrix} 0 & m_i \\ -m_i & 0 \end{pmatrix}.$$

n will always stand for the dimension of a body, m - for the number of non-zero ω_i's (i.e. for the number of two-dimensional planes in the decomposition (5)).

For a fixed λ let $A = J^2 - \lambda E$ if $\lambda \neq \infty$ or $A = E$ otherwise. By a_i denote the diagonal entries of the matrix A. Clearly $a_i = \lambda_i^2 - \lambda$ if $\lambda \neq \infty$ and $a_i = 1$ otherwise.

It will also be convenient to define the value of $\chi_i(x)$ given by (6) at the point infinity:

$$\chi_i(\infty) = \frac{1}{\omega_i^2(\lambda_{2i-1} + \lambda_{2i})^2} = \frac{1}{m_i^2}.$$

This agrees with the definition "Two parabolas intersect at infinity, if their leading coefficients are equal."

P_λ will stand for the Poisson tensor of the bracket $\{\,,\}_\lambda$ (given by (8)) at a given point.

By $\operatorname{sgrad} f$ we will mean the hamiltonian vector field generated by f:

$$\operatorname{sgrad} f = P_\infty \mathrm{d}f.$$

E_{ij} will always denote the matrix with a one in position (i, j) and zeros elsewhere.

4.4 Description of $\Lambda(M)$

Let M be a zero-rank singular point. Let us find a basis such that J is diagonal and M is block-diagonal. Introduce the following subspaces:

- $K \subset \mathfrak{so}(n)$ is generated by $E_{2i-1,2i} - E_{2i,2i-1}$ where $i = 1, \dots, m$ and all $E_{ij} - E_{ji}$ for $2m < i < j \leq n$.
- $V_{ij} \subset \mathfrak{so}(n)$ is a subspace generated by $E_{2i-1,2j-1} - E_{2j-1,2i-1}$, $E_{2i-1,2j} - E_{2j,2i-1}$, $E_{2i,2j-1} - E_{2j-1,2i}$, $E_{2i,2j} - E_{2j,2i}$.
- $W_{ij} \subset \mathfrak{so}(n)$ is a subspace generated by $E_{2i-1,j} - E_{j,2i-1}$, $E_{2i,j} - E_{j,2i}$.

We have a vector space decomposition

$$\mathfrak{so}(n) = K \oplus \bigoplus_{1 \leq i < j \leq m} V_{ij} \oplus \bigoplus_{\substack{1 \leq i \leq m, \\ 2m < j \leq n}} W_{ij}.$$

Proposition 4.1. *The space K belongs to the common kernel of all brackets of the pencil at the point M. All spaces V_{ij}, W_i are pairwise orthogonal with respect to all brackets of the pencil at M.*

Proof. This is a simple straightforward computation. $\square$

Therefore, the rank of a bracket of the pencil drops if and only if this bracket is degenerate on one of V_{ij} or W_{ij}. Let us calculate our brackets on these spaces.

Identify V_{ij} with the space of two-by-two matrices and W_{ij} with $\mathbb{R}^2$. Let $M_1, \dots M_m$ be two-by-two diagonal blocks of M. Let $A = J^2 - \lambda E$ if $\lambda \neq \infty$ and $A = E$ otherwise. Write down A as

$$A = \begin{pmatrix} A_1 & & & & & \\ & \ddots & & & & \\ & & A_m & & & \\ & & & a_{2m+1} & & \\ & & & & \ddots & \\ & & & & & a_n \end{pmatrix},$$

where A_i are two-by-two diagonal matrices and a_i are numbers.

Proposition 4.2. *The form P_λ restricted to V_{ij} has the form*

$$P_\lambda(X, Y) = 2\mathrm{Tr}\,(M_i X A_j Y^{\mathrm{t}} + M_j X^{\mathrm{t}} A_i Y).$$

The form P_λ restricted to W_{ij} has the form

$$P_\lambda(v, w) = -2a_j M_i(v, w).$$

Proof. This is a straightforward computation. $\square$

Let us now calculate P_λ on V_{ij} in coordinates. Let

$$A_s = \begin{pmatrix} a_{2s-1} & 0 \\ 0 & a_{2s} \end{pmatrix}, X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, Y = \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

Explicit calculation shows that

$$\begin{aligned} P_\lambda(X, Y) = 2(m_i a_{2j-1} c + m_j a_{2i-1} b)e + 2(m_j a_{2i} d - m_i a_{2j-1} a)g+ \\ + 2(m_i a_{2j} d - m_j a_{2i-1} a)f - 2(m_i a_{2j} b + m_j a_{2i} c)h. \end{aligned}$$

Consequently $X \in \mathrm{Ker}\, P_\lambda$ if and only if

$$\begin{cases} m_i a_{2j-1} c + m_j a_{2i-1} b = 0, \\ m_j a_{2i} d - m_i a_{2j-1} a = 0, \\ m_i a_{2j} d - m_j a_{2i-1} a = 0, \\ m_i a_{2j} b + m_j a_{2i} c = 0. \end{cases}$$

This system can be split into two two-by-two systems and the determinant of both of them equals

$$\det = m_j^2 a_{2i-1} a_{2i} - m_i^2 a_{2j-1} a_{2j}.$$

Consequently, we have proved the following

Proposition 4.3. *P_λ is degenerate on V_{ij} if and only if*

$$m_j^2 a_{2i-1} a_{2i} - m_i^2 a_{2j-1} a_{2j} = 0, \tag{10}$$

where $a_s = \lambda_s^2 - \lambda$ if $\lambda \neq \infty$ and $a_s = 1$ if $\lambda = \infty$.

The kernel in this case is given by

$$X = \begin{pmatrix} \alpha m_j a_{2i} & \beta m_i a_{2j-1}, \\ -\beta m_j a_{2i-1} & \alpha m_i a_{2j-1} \end{pmatrix},$$

where α and β are arbitrary numbers.

Now we shall study P_λ on W_{ij}.

Proposition 4.4. *P_λ is degenerate (and, consequently, zero) on W_{ij} if and only if $\lambda = \lambda_j^2$.*

The proof is straightforward.

Proposition 4.5. *The intersection of kernels of all brackets of the pencil is exactly K. For almost all brackets the kernel is exactly K.*

Proof. Indeed, only finite number of brackets are degenerate on each V_{ij} and W_{ij}. □

Now we are able to describe the *Bad* set.

Proof of Proposition 3.3. Indeed, for almost all brackets the kernel is exactly K, which means that all brackets are degenerate if and only if $\dim K > [n/2]$. This is equivalent to the condition $\dim \operatorname{Ker} M > 2$, q.e.d. □

Now we can prove the following

Proposition 4.6. *Let $M \notin Bad$. Then $\Lambda(M)$ is the set of x coordinates of the intersections on the parabolic diagram of M.*

Proof. P_λ is degenerate on V_{ij} if and only if $m_j^2 a_{2i-1}a_{2i} - m_i^2 a_{2j-1}a_{2j} = 0$. This can be rewritten as

$$\chi_i(\lambda) = \chi_j(\lambda).$$

But this means that λ is the x coordinate of the intersection point of two parabolas.

Further, P_λ is degenerate on W_{ij} if and only if $\lambda = \lambda_j^2$. But this means that λ is the x coordinate of the intersection point of the vertical line with any parabola. □

4.5 When is the pencil diagonalizable?

As a next step, we should check that the pencil is diagonalizable at M.

Proposition 4.7. *The pencil is diagonalizable at a point $M \notin Bad$ if and only if any two parabolas on the parabolic diagram of M intersect at two different points.*

Proof. Proposition 3.2 of [50] implies that the pencil is diagonalizable if and only if

$$\mathbb{C} \otimes \mathfrak{so}(n)/K = \bigoplus_{\lambda \in \Lambda(M)} \operatorname{Ker} \left(P_\lambda \mid_{\mathbb{C}\otimes\mathfrak{so}(n)/K}\right). \tag{11}$$

We have

$$\mathbb{C} \otimes \mathfrak{so}(n)/K = \bigoplus_{1\le i<j\le m} \mathbb{C} \otimes V_{ij} \oplus \bigoplus_{\substack{1\le i\le m,\\ 2m<j\le n}} \mathbb{C} \otimes W_{ij},$$

Since all the summands of this decomposition are pairwise orthogonal with respect to P_λ, (11) is satisfied if and only if

$$\mathbb{C} \otimes V_{ij} = \bigoplus_{\lambda \in \Lambda(M)} \operatorname{Ker} \left(P_\lambda \mid_{\mathbb{C}\otimes V_{ij}}\right) \text{ for } 1 \le i < j \le m, \tag{12}$$

$$\mathbb{C} \otimes W_{ij} = \bigoplus_{\lambda \in \Lambda(M)} \operatorname{Ker} \left(P_\lambda \mid_{\mathbb{C}\otimes W_{ij}}\right) \text{ for } 1 \le i \le m, 2m < j \le n. \tag{13}$$

Since there is a unique $\lambda = \lambda_j^2$ such that $\mathbb{C} \otimes W_{ij} = \operatorname{Ker}\left(P_\lambda \mid_{\mathbb{C}\otimes W_{ij}}\right)$, the relation (13) is always satisfied.

Relation (12) is satisfied if and only if the equation (10) has two distinct roots, i.e. if corresponding two parabolas are not tangent to each other, q.e.d. □

4.6 Linearization

The last step is to linearize the pencil and to check whether the linearizations are non-degenerate. We have

$$\mathfrak{so}(n) = K \oplus \bigoplus_{1\le i<j\le m} V_{ij} \oplus \bigoplus_{\substack{1\le i\le m,\\ 2m<j\le n}} W_{ij}.$$

The kernel of each $P_\lambda(M)$ can be decomposed in the following way

$$\operatorname{Ker} P_\lambda = K \oplus \bigoplus_{1\le i<j\le m} \widetilde{V}_{ij} \oplus \bigoplus_{\substack{1\le i\le m,\\ 2m<j\le n}} \widetilde{W}_{ij},$$

where $\widetilde{V}_{ij} \subset V_{ij}, \widetilde{W}_{ij} \subset W_{ij}$. For complex values of λ we have $\widetilde{V}_{ij} \subset \mathbb{C} \otimes V_{ij}$.

Proposition 4.8. *The spaces $\widetilde{V}_{ij}, \widetilde{W}_{ij}$ are invariant with respect to the adjoint operators* ad_X *in $\mathfrak{g}_\lambda$, where $X \in K$.*

Proof. Linearization $\mathfrak{g}_\lambda$ in our case is simply the stabilizer of M with respect to the bracket $[\,,]_\lambda$. Consequently, the commutator in $\mathfrak{g}_\lambda$ has the form

$$[X, Y]_\lambda = XAY - YAX,$$

where $A = J^2 - \lambda E$ for finite λ and $A = E$ for $\lambda = \infty$.

Let $X \in K, Y \in V_{ij}$. Then it is easy to see that $[X, Y]_\lambda \in V_{ij}$, which means that $\mathrm{ad}_X(V_{ij}) \subset V_{ij}$. But $\widetilde{V}_{ij} = V_{ij} \cap \operatorname{Ker} P_\lambda$, therefore $\widetilde{V}_{ij}$ is invariant. The proof for $\widetilde{W}_{ij}$ is the same. □

Now represent an element $X \in K$ as

$$X = \begin{pmatrix} 0 & x_1 & \\ -x_1 & 0 & \\ & & \ddots \end{pmatrix}.$$

Proposition 4.9. *Consider the case when $\widetilde{V}_{ij}$ is non-empty. Then the eigenvalues of* ad_X *restricted to $\widetilde{V}_{ij}$ are $\pm\nu_{ij}(X)$, where*

$$\nu_{ij}(X) = \sqrt{-\chi_i(\lambda)}(m_j x_j - m_i x_i). \tag{14}$$

Proof. This is a straightforward computation. □

Remark 4.3. If $\widetilde{V}_{ij}$ is non-empty, then $\chi_i(\lambda) = \chi_j(\lambda)$, therefore the formula (14) is actually symmetric.

Proposition 4.10. *Consider the case when $\widetilde{W}_{ij}$ is non-empty. Then the eigenvalues of* ad_X *restricted on $\widetilde{W}_{ij}$ are $\pm\mu_i(X)$, where*

$$\mu_i(X) = \sqrt{-\chi_i(\lambda)} m_i x_i.$$

Proof. This is a straightforward computation. □

Consider the following set R of linear functions on K:

$$R = \{\nu_{ij}(X) \mid i, j \text{ are such that } \widetilde{V}_{ij} \neq 0\} \cup \{\mu_i(X) \mid i \text{ is such that } \widetilde{W}_{ij} \neq 0 \text{ for some } j\}.$$

Now we see that the following is true

Proposition 4.11. *K is a diagonalizable subalgebra in $\mathfrak{g}_\lambda$. The set of roots of $\mathfrak{g}_\lambda$ with respect to K is the set $\{\pm\xi \mid \xi \in R\}$.*

Proposition 4.12. *The λ-linearization of the pencil is non-degenerate at M if and only if there are no three objects (which may be parabolas or vertical lines) on the parabolic diagram which intersect at a point with x coordinate equal to λ.*

Proof. To prove our proposition it suffices to show that the linear functions belonging to R are independent if and only if there are on three objects on the parabolic diagram which intersect at a point with x coordinate equal to λ (see Theorem 4 of [50]).

First note that up to multiplication on non-zero constant the elements of R are

$$\widetilde{\nu}_{ij}(X) = m_i x_i - m_j x_j,$$
$$\widetilde{\mu}_i(X) = m_i x_i.$$

If there are three parabolas intersecting at one point (with x coordinate equal to λ), then the spaces $\widetilde{V}_{ij}, \widetilde{V}_{jk}, \widetilde{V}_{ik}$ are non-empty for some different i, j, k. But this implies that R contains $\widetilde{\nu}_{ij}, \widetilde{\nu}_{jk}, \widetilde{\nu}_{ik}$ and since

$$\widetilde{\nu}_{ij} + \widetilde{\nu}_{jk} = \widetilde{\nu}_{ik},$$

the elements of R are not independent and the linearization is degenerate.

Now suppose that there is an intersection of two parabolas and one vertical line. Then R contains $\widetilde{\nu}_{ij}, \widetilde{\mu}_i, \widetilde{\mu}_j$ and since

$$\widetilde{\nu}_{ij} = \widetilde{\mu}_i - \widetilde{\mu}_j,$$

the elements of R are not independent and the linearization is again degenerate.

Vice versa, suppose we do not have any triple intersections. Consider some intersection point. One of the intersecting objects is necessary a parabola. Therefore R contains either some $\widetilde{\nu}_{ij} = m_i x_i - m_j x_j$ or $\widetilde{\mu}_i = m_i x_i$. The coefficient in front of x_i is non-zero in both cases. But there are no other elements of R, containing the term $m_i x_i$. Indeed, a parabola can't intersect any other object at a point with the same x coordinate. Therefore, the element of R which is given by each pairwise intersection is independent with the other elements of R, which means that R is independent and the linearization is indeed non-degenerate. □

Proposition 4.13. *Let the λ-linearization of the pencil be non-degenerate at M, where λ is real or infinite. Then the type of $Sing(\mathrm{d}_\lambda \Pi(M))$ is $(k_e, k_h, 0)$, where*

- *k_e is the number of intersections on the parabolic diagram such that their x coordinate is λ and their y coordinate is positive;*
- *k_h is the number of intersections on the parabolic diagram such that their x coordinate is λ and their y coordinate is negative.*

Proof. Indeed, each intersection of parabolas correspond to the pair of roots $\pm\nu_{ij}(X)$, where

$$\nu_{ij}(X) = \sqrt{-\chi_i(\lambda)}(m_j x_j - m_i x_i).$$

The y coordinate of the intersection point is $y = \chi_i(\lambda)$ and ν_{ij} is real if and only if this number is negative.

Intersection of a parabola with a vertical line corresponds to a pair $\pm\mu_i(X)$, where

$$\mu_i(X) = \sqrt{-\chi_i(\lambda)} m_i x_i.$$

Again, μ_i is real if and only if the y coordinate of the intersection given by $y = \chi_i(\lambda)$ is negative.

We conclude that the number of pairs of real roots equals the number of intersections in the lower half-plane, while the number of pairs of imaginary roots equals the number of intersections in the upper half-plane. Taking into account Theorem 4 of [50], this proves our proposition. □

4.7 Proof of non-degeneracy and type theorems

Proof of Theorem 7. M is non-degenerate if and only if the pencil is diagonalizable at M and all linearizations are non-degenerate (Theorem 5 of [50]). Pencil is diagonalizable if and only if any two parabolas intersect exactly at two points (Proposition 4.7). All linearizations are non-degenerate if and only if there are no multiple intersections (Proposition 4.12). But these two conditions together give exactly the condition for a parabolic diagram to be generic. □

Proof of Theorem 8. By Theorem 6 of [50], the type of a singular point M is k_e, k_h, k_f, where

$$k_e = \sum_{\lambda \in \Lambda(M) \cap \overline{\mathbb{R}}} k_e(\lambda),$$

$$k_h = \sum_{\lambda \in \Lambda(M) \cap \overline{\mathbb{R}}} k_h(\lambda),$$

$$k_f = \sum_{\lambda \in \Lambda(M) \cap \overline{\mathbb{R}}} k_f(\lambda) + \frac{1}{2} \sum_{\substack{\lambda \in \Lambda(x), \\ \operatorname{Im} \lambda > 0}} (\dim_{\mathbb{C}} \operatorname{Ker} P_\lambda - \operatorname{corank} \Pi),$$

and $(k_e(\lambda), k_h(\lambda), k_f(\lambda))$ is the type of $Sing(\mathrm{d}_\lambda \Pi(M))$.

Let λ be real (or infinite). Then $k_e(\lambda)$ is the number of intersections with $x = \lambda, y > 0$, $k_h(\lambda)$ is the number of intersections with $x = \lambda, y < 0$, and $k_f(\lambda) = 0$ (Proposition 4.13). Therefore,

$$\sum_{\lambda \in \Lambda(M) \cap \overline{\mathbb{R}}} k_e(\lambda) = \text{the number of intersections in the upper half-plane},$$

$$\sum_{\lambda \in \Lambda(M) \cap \overline{\mathbb{R}}} k_h(\lambda) = \text{the number of intersections in the lower half-plane},$$

$$\sum_{\lambda \in \Lambda(M) \cap \overline{\mathbb{R}}} k_f(\lambda) = 0.$$

If λ is complex, then

$$\frac{1}{2}(\dim_{\mathbb{C}} \operatorname{Ker} P_\lambda - \operatorname{corank} \Pi)$$

is the number of intersections with $x = \lambda$. Since we count λ with $\operatorname{Im} \lambda > 0$, this sum is one half of the total number of complex intersections. The theorem is proved. □

5 Proof of stability theorems

5.1 Stability

Proof of Theorem 2. Take a regular symplectic leaf O passing through M. The conditions of the theorem imply that the point M has pure elliptic type on O. Therefore, there exists an integral f such that

$$f(M) = 0,$$
$$\mathrm{d}(f\mid_O)(M) = 0$$

and the Hessian of $f\mid_O$ is positive definite at M.

Since O is regular there exists a coordinate system $x_1, \ldots x_N$ in the neighborhood of M such that

$$O = \{x_i = 0, i = 1, \ldots, k\}.$$

Now it is easy to see that $F = f^2 + \sum_{i=1}^{k} x_i^2$ is a Lyapunov function. □

5.2 Non-resonancy

To prove instability theorems, we need to check that our system is non-resonant, i.e. that the trajectories of the system are dense on almost all Liouville tori (see [30] for the precise definition of a non-resonant integrable system). This seems to be the most tricky part of the whole paper.

Theorem 9. *The system of Euler-Arnold equations is non-resonant, i.e. the trajectories of this system are dense on almost all Liouville tori.*

First we will prove several preliminary statements.

Proposition 5.1. *Let M be a regular equilibrium such that the parabolic diagram of M is generic and all eigenvalues of M are distinct. Then the eigenvalues of the linearization of the Euler-Arnold vector field at M are $\pm\sigma_{ij}^{1,2}$, where*

$$\sigma_{ij}^{1,2} = \frac{1}{\sqrt{-\chi_i(x_{ij}^{1,2})}}\left(\frac{x_{ij}^{1,2} + \lambda_{2i-1}\lambda_{2i}}{\lambda_{2i-1}+\lambda_{2i}} - \frac{x_{ij}^{1,2} + \lambda_{2j-1}\lambda_{2j}}{\lambda_{2j-1}+\lambda_{2j}}\right),$$

where $1 < i < j \le [n/2]$ and $x_{ij}^{1,2}$ are two roots of the equation $\chi_i(x) = \chi_j(x)$.

If the dimension is odd, there are also eigenvalues $\pm\tau_i$, where

$$\tau_i = \frac{1}{\sqrt{-\chi_i(\lambda_n^2)}}\left(\frac{\lambda_n^2 + \lambda_{2i-1}\lambda_{2i}}{\lambda_{2i-1}+\lambda_{2i}} - \lambda_n\right),$$

where $1 < i \le [n/2]$.

Proof. Instead of the linearization of $\operatorname{sgrad} H$, we can consider the operator $D = D_H P_\infty$ (see Section 4 of [50]). Since $P_\lambda \mathrm{d}H_\lambda = P_\infty \mathrm{d}H$, we have $D = D_H P_\infty = D_{H_\lambda} P_\lambda$.

Since the parabolic diagram of M is generic, the pencil is diagonalizable at M, and there is a decomposition

$$\mathbb{C}\otimes\mathfrak{so}(n)/K = \bigoplus_{\lambda\in\Lambda(M)} \mathbb{C}\otimes \operatorname{Ker} P_\lambda(M)/K, \tag{15}$$

where K is the common kernel of regular brackets of the pencil at M (see Proposition 4.1 of [50]).

Since $\operatorname{Ker} P_\lambda$ is invariant with respect to $D_{H_\lambda} P_\lambda$, the decomposition (15) is invariant with respect to the operator D. Now note that

$$D\mid_{\operatorname{Ker} P_\lambda} = \mathrm{ad}_{\mathrm{d}H_\lambda},$$

where ad is the adjoint operator in $\operatorname{Ker} P_\lambda$. Now, applying Propositions 4.9, 4.10 and taking into account the formula (9), we get the formula for the eigenvalues. □

Proposition 5.2. *Let $\lambda_{2i-1}\lambda_{2i} \ne \lambda_{2j-1}\lambda_{2j}$ for all i,j. Then $\sigma_{ij}^{1,2}$ and τ_i are linearly independent as functions of M.*

Proof. Suppose that

$$\sum_{1<i<j\le[n/2]} (a_{ij}^1\sigma_{ij}^1 + a_{ij}^2\sigma_{ij}^2) + \sum_{1<i\le[n/2]} b_i\tau_i = 0.$$

Fix k and choose those members which do not depend on m_k. Their sum does not depend on m_k, therefore the sum of all other members

$$S_k = \sum_{1<i<k} (a_{ik}^1\sigma_{ik}^1 + a_{ik}^2\sigma_{ik}^2) + \sum_{k<j\le[n/2]} (a_{kj}^1\sigma_{kj}^1 + a_{kj}^2\sigma_{kj}^2) + b_k\tau_k \tag{16}$$

does not depend on m_k as well. For simplicity denote

$$\sigma_{jk} = -\sigma_{kj}, a_{jk} = -a_{kj}$$

and rewrite (16) as

$$S_k = \sum_{i \neq k} (a^1_{ik}\sigma^1_{ik} + a^2_{ik}\sigma^2_{ik}) + b_k\tau_k. \tag{17}$$

Let m_k tend to zero. It is easy to see that

$$\lim_{m_k \to 0} \sigma^{1,2}_{ik} = 0, \lim_{m_k \to 0} \tau_k = 0.$$

Consequently,

$$\lim_{m_k \to 0} S_k = 0.$$

But S_k does not depend on m_k which means that $S_k = 0$.

Now fix $l \neq k$. Again, there are two summands in (17), which depend on m_l. Their sum

$$S_{lk} = a^1_{lk}\sigma^1_{lk} + a^2_{lk}\sigma^2_{lk}$$

must not depend on m_l. But this sum tends to 0 as $m_l \to 0$, therefore $S_{lk} = 0$. But

$$\lim_{m_k \to \infty} \sigma^1_{lk} = Am_l, \lim_{m_k \to \infty} \sigma^2_{lk} = Bm_l,$$

$$A = \pm\frac{1}{\lambda_{2l-1} + \lambda_{2l}}\sqrt{-\frac{(\lambda_{2k-1} - \lambda_{2l-1})(\lambda_{2k-1} - \lambda_{2l})}{(\lambda_{2k-1} + \lambda_{2l-1})(\lambda_{2k-1} + \lambda_{2l})}},$$
$$B = \pm\frac{1}{\lambda_{2l-1} + \lambda_{2l}}\sqrt{-\frac{(\lambda_{2k} - \lambda_{2l-1})(\lambda_{2k} - \lambda_{2l})}{(\lambda_{2k} + \lambda_{2l-1})(\lambda_{2k} + \lambda_{2l})}},$$

where the signs depend on the relative positions of λ's. Consequently,

$$a^1_{lk}A + a^2_{lk}B = 0. \tag{18}$$

On the other hand,

$$\lim_{m_l \to \infty} \sigma^1_{lk} = Cm_k, \lim_{m_l \to \infty} \sigma^2_{lk} = Dm_k,$$

$$C = \pm\frac{1}{\lambda_{2k-1} + \lambda_{2k}}\sqrt{-\frac{(\lambda_{2l-1} - \lambda_{2k-1})(\lambda_{2l-1} - \lambda_{2k})}{(\lambda_{2l-1} + \lambda_{2k-1})(\lambda_{2l-1} + \lambda_{2k})}},$$
$$D = \pm\frac{1}{\lambda_{2k-1} + \lambda_{2k}}\sqrt{-\frac{(\lambda_{2l} - \lambda_{2k-1})(\lambda_{2l} - \lambda_{2k})}{(\lambda_{2l} + \lambda_{2k-1})(\lambda_{2l} + \lambda_{2k})}}.$$

Consequently,

$$a^1_{lk}C + a^2_{lk}D = 0. \tag{19}$$

It is easy to see that $AD - BC = 0$ if and only if $\lambda_{2l-1}\lambda_{2l} = \lambda_{2k-1}\lambda_{2k}$. But we assumed that this equality is not satisfied. Therefore (18) and (19) imply that $a^1_{lk} = a^2_{lk} = 0$. Since k and l were arbitrary, all coefficients $a^{1,2}_{ij}$ vanish. But this implies that b_i vanish as well and our functions are linearly independent, q.e.d. $\square$

Remark 5.1. It is easy to see that if $\lambda_{2i-1}\lambda_{2i} = \lambda_{2j-1}\lambda_{2j}$ for some i, j, then $\sigma^1_{ij} = \sigma^2_{ij}$, which means that the eigenvalues are not linearly independent.

Now the proof of Theorem 9 follows from the following simple lemma:

Lemma 5.1. *Let $f_1, \dots f_n$ be continuous functions on a manifold N. Suppose that these functions are linearly independent on any open subset $V \subset N$. Then their values are independent over $\mathbb{Z}$ almost everywhere on N.*

Proof of Theorem 9. Let us consider an equilibrium M such that $\lambda_1 < \lambda_2 < \dots$ and the parabolic diagram is generic. In the neighborhood of M the eigenvalues of $\operatorname{sgrad} H$ are linearly independent. Then, by the previous lemma, they are independent over $\mathbb{Z}$ almost everywhere. Take a point M_1 such that they are independent. We claim that our system is non-resonant in the neighborhood of M_1. Indeed, M_1 is an elliptic point and the Liouville foliation on any regular symplectic leaf passing through M_1 is locally given by the functions

$$s_1 = p_1^2 + q_1^2, \dots, s_m = p_m^2 + q_m^2$$

in some symplectic coordinates p, q (Eliasson theorem [28]).

The functions s_i are action variables and the rotation numbers are given by

$$c_i(s_1, \dots, s_m) = \frac{\partial H}{\partial s_i}.$$

It is easy to see that $c_i(0)$ are exactly eigenfrequencies of $\operatorname{sgrad} H$. They are independent over $\mathbb{Z}$. But this implies that c_i are independent on almost all tori in the neighborhood of M_1 and our system is non-resonant. By analyticity, it is non-resonant everywhere. $\square$

5.3 Instability

To prove the instability theorem, we will need the following simple

Lemma 5.2 (About unstable cone)**.** *Let a vector field v on a manifold N vanish at point x_0. Suppose that the linearization of v at x_0 has an eigenvalue with a positive real part. Then there is an open subset $K \subset N$ such that*

1. *There exists $\delta > 0$ such that all trajectories starting at K leave $U_\delta(x_0)$.*
2. *The intersection of K with any open neighborhood of x_0 has non-empty interior.*

Corollary 5.1. *Let $v = \operatorname{sgrad} H$ be a non-resonant integrable hamiltonian system and x be an equilibrium point of it. Suppose that there exists an integral f such that $\operatorname{sgrad} f(x) = 0$ and the linearization of $\operatorname{sgrad} f$ at x has an eigenvalue with non-zero real part. Then x is an unstable equilibrium for $\operatorname{sgrad} H$.*

Proof. Since the linearization of $\operatorname{sgrad} f$ at x has an eigenvalue with non-zero real part, it has an eigenvalue with positive real part. Therefore, we can find a set K from the lemma. For any ε the intersection $U_\varepsilon(x) \cap K$ has non-empty interior. Therefore, we can find a non-resonant torus passing through $U_\varepsilon(x) \cap K$. The trajectory of $\operatorname{sgrad} f$ lies on this torus, therefore this torus will leave $U_\delta(x)$. But since the torus is non-resonant, all trajectories of $\operatorname{sgrad} H$ are dense on it and will leave $U_\delta(x)$ as well. Therefore, x is an unstable equilibrium, q.e.d. $\square$

Proof of Theorem 3. By Proposition 4.4 of [50] the following sets of operators are equal

$$\{D_f P_\infty\, |_{\operatorname{Ker} P_\lambda}\}_{f \in \mathcal{F}, \mathrm{d}f \in \operatorname{Ker} P_\infty} = \{\operatorname{ad}_\xi\}_{\xi \in K},$$

where ad_ξ is the adjoint operator in $\mathfrak{g}_\lambda$.

Conditions of the theorem imply that for some λ there is $\xi \in K$ such that the operator ad_ξ has an eigenvalue with non-zero real part (see the formulas for the eigenvalues given by Propositions 4.9, 4.10). Therefore, there exists an integral f such that $D_f P_\infty$ also has such an eigenvalue. But $D_f P_\infty$ is dual to the linearization of $\operatorname{sgrad} f$. Now it suffices to apply Corollary 5.1. $\square$

5.4 Instability of exotic equilibria

Now we shall prove that all exotic equilibria are unstable.

Proposition 5.3. *Let M be an exotic equilibrium. Then there is λ such that* $\operatorname{sgrad} H_\lambda \neq 0$, *where*

$$H_\lambda = -\frac{1}{2}\langle (J+\sqrt{\lambda}E)^{-1}\Omega(J+\sqrt{\lambda}E)^{-1}, M\rangle.$$

Proof. The vector field $\operatorname{sgrad} H_\lambda$ has the form

$$\dot{M} = [(J+\nu E)^{-1}\Omega(J+\nu E)^{-1}, M], \text{ where } \nu = \sqrt{\lambda}.$$

Suppose that M is an equilibrium point for $\operatorname{sgrad} H_\lambda$ for all λ. Then

$$\begin{aligned} 0 = [(J+\nu E)^{-1}\Omega(J+\nu E)^{-1}, J\Omega+\Omega J] = (J+\nu E)^{-1}\Omega^2 - \Omega^2(J+\nu E)^{-1} + \\ + (J+\nu E)^{-1}\Omega(J+\nu E)^{-1}\Omega(J-\nu E) - (J-\nu E)\Omega(J+\nu E)^{-1}\Omega(J+\nu E)^{-1}. \end{aligned}$$

Since M is an equilibrium point of the body, Ω^2 commutes with J. Therefore, it also commutes with $(J+\nu E)^{-1}$. Consequently,

$$(J+\nu E)^{-1}\Omega(J+\nu E)^{-1}\Omega(J-\nu E) = (J-\nu E)\Omega(J+\nu E)^{-1}\Omega(J+\nu E)^{-1}.$$

But this equality means that the matrix

$$(J+\nu E)^{-1}\Omega(J+\nu E)^{-1}\Omega(J-\nu E)$$

is symmetric. Let us denote by ω_{ij} the entries of the matrix Ω. λ_i are, as usual, diagonal entries of J. Then the symmetry condition can be written as:

$$\frac{\lambda_k-\nu}{\lambda_i+\nu}\sum_j \frac{\omega_{ij}\omega_{jk}}{\lambda_j+\nu} = \frac{\lambda_i-\nu}{\lambda_k+\nu}\sum_j \frac{\omega_{ij}\omega_{jk}}{\lambda_j+\nu} \text{ for all } i,k.$$

Since all eigenvalues of J are distinct, this implies

$$\sum_j \frac{\omega_{ij}\omega_{jk}}{\lambda_j+\nu} = 0 \text{ for all } i \neq k.$$

This equation should be satisfied for all ν. But this means that

$$\omega_{ij}\omega_{jk} = 0$$

for all j and all distinct i,k. Therefore, we can bring Ω to the block-diagonal form with two-by-two blocks on the diagonal by permuting basis vectors. Such a permutation will preserve diagonal form of J. Consequently, M is not exotic, but regular. Contradiction. $\square$

Proof of Theorem 4. By the previous Proposition we can find an integral f such that

$$\operatorname{sgrad} f(M) \neq 0$$

for a given exotic equilibrium M. Obviously, the trajectories of $\operatorname{sgrad} f$ leave sufficiently small neighborhood of M. Therefore, Liouville tori leave this neighborhood as well. Since our system is non-resonant, it's trajectories are dense on most Liouville tori and will also leave the neighborhood, q.e.d. $\square$

Remark 5.2. We used Proposition 5.3 to show that exotic equilibria are not rank zero singular points. For equilibria not belonging to the set *Bad* this follows automatically from Theorem 6.

Remark 5.3. Since $\operatorname{sgrad} H_\lambda(M) \neq 0$ for any exotic equilibrium M and some value λ, exotic equilibria are not isolated on the symplectic leaves of the $\mathfrak{so}(n)$-bracket, but form smooth families. The dimension of such a family essentially depends on the sizes of the blocks A_i, entering formula (4).

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